

Bounds on the Average Sensitivity of Nested Canalizing Functions

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Abstract

Nested canalizing Boolean functions play an important role in biological motivated regulative networks but also in signal processing, such as in describing stack filters. It has been conjectured that this class of functions has a stabilizing effect on the network dynamics. It is well known that the average sensitivity plays a central role for the stability of (Random) Boolean networks. Here we prove a tight upper bound on the average sensitivity for nested canalizing functions in dependence of the number of relevant input variables. We further show that it is smaller than $\frac{4}{3}$ as conjectured in literature. This shows that a large number of functions appearing in biological networks belong to a class that has a very low average sensitivity, which is even close to a tight lower bound.

1. Introduction

Boolean networks play an important role in modeling and understanding signal transduction and regulatory networks. These networks have been widely studied with focus on many facets, e.g. [1, 2, 3]. One line of research focuses on the dynamical stability of randomly created networks. For example, random Boolean networks tend to be unstable if the functions are chosen from all possible Boolean functions and the average number of variables (average *in-degree*) is larger than two [4]. This can be attributed to the fact that expected *average sensitivity* of the random functions, which is an appropriate measure for the stability of random Boolean networks [5, 6], is too large.

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If only functions from certain classes are chosen, stable behaviour can be achieved for higher in-degrees. For instance, canalizing and nested canalizing functions, introduced in [7, 8], have been conjectured [9] to have a stabilizing effect on network dynamics. In [10] it has been shown that stable behaviour is possible even for networks with in-degree. Interestingly, studies of regulatory network models have shown that a large number of their functions are canalizing [11, 12, 13, 14, 15, 16]. Canalizing functions are also important for the construction of stack filters used in signal processing [17].

A function is *canalizing* in a variable, if its output is constant when this variable is set to its *canalizing* value. Nested canalizing functions are canalizing functions, whose restriction to the non-canalizing value is again a canalizing function and so on (a precise definition is given later). In this paper we analyze nested canalizing functions and we focus in particular on their average sensitivity. The notion of sensitivity was first introduced by Cook et al. [18]. Later it was applied to Boolean functions [19] and can be viewed as a metric for the influence of a random permutation of the input variables on the output of the function. Since then the average sensitivity has been studied widely. In [20] the average sensitivity in the context of monotone Boolean functions was investigated and an upper bound for locally monotone functions was presented in [15]. Here we give a tight upper-bound on the average sensitivity of nested canalizing functions with different numbers of variables. This also shows that the average sensitivity is always smaller than $\frac{4}{3}$ as conjectured in [21]. We further give a recursive expression of the average sensitivity and the bias of this class of functions. Finally we will discuss and compare our new bounds and some old bounds.

Our main tool is the Fourier analysis [22, 23] of Boolean functions, which is introduced in Section 2. There we also address further concepts needed, such as restrictions of Boolean functions. In Section 3 spectral properties of canalizing and nested canalizing functions are broached. Additionally we discuss functions, in which all variables are most dominant, as they turn out later to minimize the average sensitivity. In Section 4 the new bounds on the average sensitivity are presented based on a recursive expression of the average sensitivity of nested canalizing functions. We conclude then with a discussion of the obtained results and some final remarks.

2. Notation, Basic Definitions and Fourier Analysis of Boolean Functions

A Boolean function (BF) $f \in \mathcal{F}_n = \{f : \Omega^n \rightarrow \Omega\}$ with $\Omega = \{-1, +1\}$ maps n -ary input tuples to a binary output. In general not all input variables have an impact on

the output, i.e. are relevant.

Definition 1. [21] *A variable i is relevant to a BF f , if there exists an $\mathbf{x} \in \Omega^n$ such that*

$$f(\mathbf{x}) \neq f(\mathbf{x} \oplus e_i),$$

where $x \oplus e_i$ is the vector obtained from \mathbf{x} by flipping its i -th entry.

Further $\text{rel}(f)$ is the set containing all relevant variables of f .

2.1. Fourier Analysis of Boolean Functions

In this section we will recall some basic concepts of Fourier analysis and some results concerning restrictions of BF as shown in [24]. Let us consider $\mathbf{x} = (x_1, x_2, \dots, x_n)$ as an instance of a uniform distributed random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$, i.e., its probability density functions can be written as

$$\Pr[\mathbf{X} = \mathbf{x}] = \frac{1}{2^n}.$$

We first define the basis of the Fourier transform as the set of monomials $\chi_U(\mathbf{x})$ given by all $U \subseteq [n] = \{1, 2, \dots, n\}$, where

$$\chi_U(\mathbf{x}) = \prod_{i \in U} x_i. \quad (1)$$

For $U = \emptyset$ we set $\chi_{\emptyset}(\mathbf{x}) = 1$.

It is well known that any BF f can be expressed by the following sum, called Fourier-expansion [22, 23],

$$f(\mathbf{x}) = \sum_{U \subseteq [n]} \hat{f}(U) \cdot \chi_U(\mathbf{x}),$$

where $\hat{f}(U)$ are the Fourier coefficients. The Fourier coefficients can be recovered by

$$\hat{f}(U) = 2^{-n} \sum_{\mathbf{x}} f(\mathbf{x}) \cdot \chi_U(\mathbf{x}). \quad (2)$$

Let $A \subset U$ and $\bar{A} = U \setminus A$, then

$$\chi_U(\mathbf{x}) = \chi_A(\mathbf{x}) \cdot \chi_{\bar{A}}(\mathbf{x}),$$

which directly follows from the definition of χ_U (Eq. (1)).

2.2. Restrictions of Boolean Functions

If we restrict f , i.e. if we set the i -th input variable of f to some constant $a \in \{-1, +1\}$, we express the obtained new function by $f^{(i,a)} \in \mathcal{F}_n$. Every BF can be decomposed in two unique restricted functions for each relevant variable, as stated in the following proposition:

Proposition 1. *For any $f \in \mathcal{F}_n$ and each $i \in [n]$ there exist unique functions $f^{(i,+)}, f^{(i,-)} \in \mathcal{F}_n$, with $i \notin \text{rel}(f^{(i,+)})$ and $i \notin \text{rel}(f^{(i,-)})$, such that*

$$f = g^{(i,+)} f^{(i,+)} + g^{(i,-)} f^{(i,-)},$$

where the functions $g^{(i,+)}, g^{(i,-)} \in \mathcal{F}_n$ are given by

$$g^{(i,+)}(\mathbf{x}) = \begin{cases} 1 & \text{if } x_i = 1 \\ 0 & \text{else} \end{cases} \quad \text{and} \quad g^{(i,-)}(\mathbf{x}) = \begin{cases} 1 & \text{if } x_i = -1 \\ 0 & \text{else} \end{cases}.$$

The Fourier coefficients of these restricted functions can be derived as stated in the following proposition, the proof and an extension for product distributed input variables can be found in [24].

Proposition 2. [24] *Let f be a BF in n uniformly distributed variables. Consider the restricted function obtained by setting $x_i = a_i$, then*

$$\hat{f}^{(i,a_i)}(U) = \hat{f}(U) + a_i \cdot \hat{f}(U \cup \{i\})$$

where $U \subseteq [n] \setminus \{i\}$.

The reverse relation, i.e. the composition of a BF by two restricted functions, is described in terms of Fourier coefficients by the following proposition. The proof can again be found in [24].

Proposition 3. [24] *The Fourier coefficients of any BF f with uniform distributed input variables can be composed of the coefficients of its two restricted functions $f^{(i,-1)}$ and $f^{(i,+1)}$*

$$\hat{f}(U) = \frac{1}{2} \left(\hat{f}^{(i,+)}(U \setminus \{i\}) + (-1)^{|U \cap \{i\}|} \hat{f}^{(i,-)}(U \setminus \{i\}) \right),$$

or

$$\hat{2}f(U) = \begin{cases} \hat{f}^{(i,+)}(U \setminus \{i\}) + \hat{f}^{(i,-)}(U \setminus \{i\}) & \text{if } i \in U \\ \hat{f}^{(i,+)}(U) - \hat{f}^{(i,-)}(U) & \text{if } i \notin U \end{cases}.$$

The zero coefficient $\hat{f}(\emptyset)$ plays an import role in the analysis of BFs. In the uniform case it corresponds to the bias of the function f , where bias is defined as the probability that $f(x)$ is -1 . Next, we show that it can be easily composed by the zero coefficients of the restricted functions. We will need this later to show the recursive behavior of nested canalizing functions.

Corollary 1. *The zero coefficient of any Boolean function f can be written as:*

$$\hat{f}(\emptyset) = \frac{1}{2}\hat{f}^{(i,+)}(\emptyset) + \frac{1}{2}\hat{f}^{(i,-)}(\emptyset), \quad (3)$$

where $i \in [n]$ is the index of some variable.

Proof. Follows directly from Proposition 3. \square

If we restrict a function to more than one variable, namely to a set of variables K , we denote the restricted function with $f^{(K,\mathbf{a})}$, where \mathbf{a} is a vector containing the values to which the functions is restricted. The Fourier coefficients are then given by the following proposition, which again can be found in [24].

Proposition 4. [24] *Let f be a Boolean function and $\hat{f}(U)$ its Fourier coefficients. Furthermore, let K be a set containing the indices i of the input variables x_i , which are fixed to certain values a_i . The Fourier coefficients of the restricted function are then given as:*

$$\hat{f}^{(K,\mathbf{a})}(U) = \sum_{S \subseteq K} \left(\Phi_S(\mathbf{a}) \cdot \hat{f}(U \cup S) \right),$$

where \mathbf{a} is a vector containing all $a_i, i \in K$.

3. Nested Canalizing Functions

3.1. General

To define nested canalizing functions (NCF) we first need to look at canalizing functions:

Definition 2. *A BF f is called $\langle i : a : b \rangle$ canalizing if there exists a canalizing variable x_i and a Boolean value $a \in \{-1, +1\}$ such that the function*

$$f(\mathbf{x}|_{x_i=a}) = f^{(i,a)}(\mathbf{x}) = b, \quad (4)$$

for all $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$, where $b \in \{-1, +1\}$ is a constant.

Hence, f is canalizing in variable i if and only if fractionation according to Proposition 1 results in either $f^{(i,+)}$ or $f^{(i,-)}$ being a constant function.

As shown in [24] the Fourier coefficients then fulfill the following condition:

$$\hat{f}(\emptyset) + a_i \cdot \hat{f}(\{i\}) = b_i. \quad (5)$$

A NCF can be described recursively as a canalizing function, whose restriction is again nested canalizing or more formally:

Definition 3. For $k = 1$ and $k = 0$ any BF with $k \leq n$ relevant variables is a NCF. For $k > 1$ a BF is a NCF iff there exists at least one variable i and two constants $\alpha_i, \beta_i \in \{+1, -1\}$ such that $f^{(i, \alpha_i)} = \beta_i$ and $f^{(i, -\alpha_i)}$ is a NCF with $k - 1$ relevant variables.

Further let $x_{\pi(1)}, \dots, x_{\pi(k)}$ be the variable order for which a NCF fulfills the properties from this definition, then we call, according to [21], such a function $\{\pi : \alpha : \beta\}$ nested canalizing.

This condition can be easily expressed using the Fourier representations: [24] f is $\{\pi : \alpha : \beta\}$ nested canalizing, if for all $j \in \{1, \dots, k\}$

$$\sum_{S \subseteq [j]} \left(\alpha_j^{|S \cap \{j\}|} \cdot \chi_{S \setminus \{j\}}(\bar{\alpha}) \cdot \hat{f}(\tilde{S}) \right) = \beta_j,$$

where $\bar{\alpha}$ is a vector containing all negated α_i , i.e. $\bar{\alpha}_i = -\alpha_i$ and \tilde{S} is a set which is retrieved by applying the permutation π to the elements of S .

Further we may want to add an example: Let f be $\{\pi : \alpha : \beta\}$ NCF with $k = 2$ relevant variables and $\pi = id$, that is $\tilde{S} = \pi(S) = S$, then we can write:

$$\begin{aligned} \beta_1 &= \hat{f}(\emptyset) + \alpha_1 \hat{f}(\{1\}) \\ \beta_2 &= \hat{f}(\emptyset) - \alpha_1 \hat{f}(\{1\}) + \alpha_2 \hat{f}(\{2\}) - \alpha_1 \alpha_2 \hat{f}(\{1, 2\}). \end{aligned}$$

3.2. Properties of Nested Canalizing Functions

In this section we state some properties of NCF. First we address most dominant variables, which are defined as:

Definition 4. According to [21] we call a variable i a most dominant variable of f if there exists at least one variable order $\pi = (i, \dots)$, for which f is $\{\pi : \alpha : \beta\}$ canalizing.

The set of most dominant variables has an impact on a number of Fourier coefficients, which is summarized in the following proposition.

Proposition 5. *Let K be the set of most dominant variables of a $\{\pi : \alpha : \beta\}$ NCF f with uniform distributed input variables. Then the absolute values of the corresponding Fourier coefficients are all equal the same constant $c > 0$, i.e.,*

$$\left| \hat{f}(U) \right| = c \quad \forall U \subseteq K, U \neq \{\emptyset\},$$

or, more general,

$$\alpha_j \cdot \chi_{U \setminus \{j\}}(\bar{\alpha}) \cdot \hat{f}(U) = c \quad \forall U \subseteq K, U \neq \{\emptyset\} \text{ and } \forall j \in K. \quad (6)$$

Further the absolute value of the zero coefficient, $\hat{f}(\emptyset)$, is $1 - c$ and the sign is given by b , i.e.,

$$b = \text{sgn} \left(\hat{f}(\emptyset) \right),$$

and

$$\beta_i = b \quad \forall i \in K.$$

The proof can be found in Appendix A.

For the special case, in which all variables are most dominant, we derive the following two corollaries:

Corollary 2. *Let f be a $\{\pi : \alpha : \beta\}$ NCF with n uniform distributed and k relevant input variables. All variables are most canalizing if and only if the Fourier coefficients fulfill the following conditions,*

$$\alpha_j \cdot \left(\prod_{i \in S, i \neq j} \bar{\alpha}_i \right) \hat{f}(S) = c \quad \forall S \subseteq [n], S \neq \{\emptyset\} \text{ and } \forall j \in S \quad (7)$$

and hence

$$\begin{aligned} \left| \hat{f}(S) \right| &= c & \forall S \subseteq K, S \neq \{\emptyset\}, \\ \left| \hat{f}(\emptyset) \right| &= 1 - c \end{aligned} \quad (8)$$

with

$$c = 2^{-(k-1)}. \quad (9)$$

Proof. Eq. (7) and (8) follow directly from Proposition 5, while Eq. (9) follows from Parsevals theorem.

□

Corollary 3. *Let f be a $\{\pi : \alpha : \beta\}$ NCF with $k > 1$ uniform distributed and relevant input variables. All variables are most canalizing and $\beta_i = b, \forall i \in \{1, \dots, k\}$. All such NCFs are completely described by α and b . and hence there are $2^{(k+1)}$ such functions.*

Proof. The proof follows directly from the previous Corollary. □

As mentioned before, the zero coefficient plays an important role. Interestingly, we can describe the zero coefficients for NCFs in a recursive manner:

Proposition 6. *The zero coefficient of a $\{\pi : \alpha : \beta\}$ NCF f can be recursively written as :*

$$\hat{f}(\emptyset) = \frac{1}{2} \hat{f}^{(\pi_i, \alpha_i)}(\emptyset) + \frac{1}{2} \beta_i.$$

Proof. Follows directly from Corollary 1. □

Further, the zero coefficient is upper-bounded as shown in the following proposition:

Proposition 7. *The absolute value of zero coefficient of a $\{\pi : \alpha : \beta\}$ NCF f with uniform distributed input variables can be bounded as:*

$$\frac{1}{3} \left(\frac{1}{2^{k-1}} (-1)^k + 1 \right) \leq |\hat{f}(\emptyset)| \leq 1 - \frac{1}{2^{k-1}},$$

where $k = \text{rel}(f)$ is the number of relevant variables.

Proof. First, we prove the right hand side: Using the triangle inequality we get from Proposition 6:

$$|\hat{f}(\emptyset)| \leq \frac{1}{2} |\hat{f}^{(\pi_i, \alpha_i)}(\emptyset)| + \frac{1}{2}.$$

Obviously the zero coefficient of a function with only one relevant variable i is zero. The proposition now follows using induction. The left hand side can be easily shown using the inverse triangle inequality and induction. □

As seen in Corollary 2, a NCF, whose variables are most dominant, fulfills the upper-bound with equality. Further, it can be easily seen that NCFs with alternating β_i , i.e., with $\beta = (-1, +1, -1, +1, \dots)$ or $\beta = (+1, -1, +1, -1, \dots)$ fulfill the lower-bound with equality.

4. Average Sensitivity

4.1. Definition

The average sensitivity (as) [19] is a measure to quantify the influence of random perturbations of the inputs of Boolean functions. It is defined as the sum of the influences of the inputs of the function, which is defined as the probability of a change of the function's output if input i is flipped:

Definition 5. ([25, 26]) Define the influence of variable i on the function f as

$$I_i(f) = \Pr[f(\mathbf{X}) \neq f(\mathbf{X} \oplus e_i)].$$

The influence can be related to the Fourier spectra as follows [27]:

$$I_i(f) = \sum_{S \subseteq [n]: i \in S} \hat{f}(S)^2.$$

The average sensitivity is defined as the sum of the influences of all input variables of f .

Definition 6. ([25, 19]) The average sensitivity of f to all input variables is defined as

$$as(f) = \sum_{i \in [n]} I_i(f).$$

Consequently the average sensitivity can also be expressed in terms of the Fourier coefficients [25]:

$$as(f) = \sum_{S \subseteq [n], S \neq \emptyset} \hat{f}(S)^2 |S|. \quad (10)$$

4.2. Restricted Functions

To investigate the behavior of the average sensitivity of restricted functions we first need to define the function $\xi : \mathcal{F}_n \times \mathcal{F}_n \rightarrow \mathbb{R}$ by

$$\xi(f, g) = \frac{1}{2} \left(1 - \sum_{U \subseteq [n]} \hat{f}(U) \hat{g}(U) \right). \quad (11)$$

We can then state the following theorem, which shows the relation between the average sensitivity of a BF and the average sensitivity of its two restricted functions.

Theorem 1. *Let $f^{(i,+)}$, $f^{(i,-)}$ be the restrictions of f to some relevant variable i of f . Then*

$$as(f) = \frac{1}{2} as(f^{(i,+)}) + \frac{1}{2} as(f^{(i,-)}) + \xi(f^{(i,+)}, f^{(i,-)})$$

The proof can be found in Appendix B. For NCFs we obtain then:

Corollary 4. *The average sensitivity of a $\{\pi : \alpha : \beta\}$ NCF can be recursively described as:*

$$as(f) = \frac{1}{2} \left(as(f^{(\pi_i, \alpha_i)}) + 1 - \hat{f}^{(\pi_i, \alpha_i)}(\emptyset) \beta_i \right). \quad (12)$$

In [21] an upper-bound on the average sensitivity of NCF has been conjectured. In the following theorem, we proof this conjecture to be correct.

Theorem 2. *The average sensitivity of a NCF with $k = rel(f)$ relevant and uniform distributed variables is bounded by*

$$\frac{k}{2^{k-1}} \leq as(f) \leq \frac{4}{3} - 2^{-k} - \frac{1}{3} \cdot 2^{-k} \cdot (-1)^k. \quad (13)$$

The bounds are tight.

The proof can be found in Appendix C. The tightness is shown in Corollaries 6 and 7. We can generalize the upper bound of Theorem 2 as:

Corollary 5. *The average sensitivity of a NCF with $k = rel(f)$ relevant and uniform distributed variables is upper-bounded by*

$$as(f) \leq \frac{4}{3}.$$

Corollary 6. *Let f be a NCF, whose variables are all most dominant, then f fulfills the left-hand-side of Theorem 2 with equality.*

Proof. We start from Corollary 4, using the fact that in this case $|\hat{f}(\emptyset)| = 1 - \frac{1}{2^{k-1}}$ and all $\beta_i = sgn(\hat{f}(\emptyset))$, we get:

$$as(f) = \frac{1}{2} \left(as(f^{(\pi_i, \alpha_i)}) + 1 - \left(1 - \frac{1}{2^{k-2}}\right) \right) \quad (14)$$

$$= \frac{1}{2} \left(as(f^{(\pi_i, \alpha_i)}) + \frac{1}{2^{k-2}} \right). \quad (15)$$

Since $\text{as}(f)$ has k relevant variables, while $\text{as}(f^{(\pi_i, \alpha_i)})$ depends only on $k - 1$ relevant variables, we can also express Eq. (14) as:

$$\text{as}(k) = \frac{1}{2} \left(\text{as}(k-1) + \frac{1}{2^{k-2}} \right).$$

The proof can now be concluded by solving this recursion using induction. \square

Corollary 7. *Let f be a NCF with alternating β_i , i.e. $\beta = (-1, +1, -1, +1, \dots)$ or $\beta = (+1, -1, +1, -1, \dots)$. Then f fulfills the right-hand-side of Theorem 2 with equality.*

Proof. According to the proof of the previous corollary we start from Corollary 4 and use $|\hat{f}(\emptyset)| = \frac{1}{3} \left(\frac{1}{2^{k-1}}(-1)^k + 1 \right)$, the poof is concluded by solving the recursion. \square

The last two corollaries showed that the maximal average sensitivity is achieved if the bias, i.e. the zero coefficient, is minimized, and vice versa. Next, we derive a common bound as given in the following proposition:

Proposition 8. *Let f be any NCF and $\hat{f}(\emptyset)$ its zero coefficient, then*

$$\text{as}(f) + |\hat{f}(\emptyset)| \leq \frac{5}{3}.$$

Proof. Combining Corollaries 4 and 5, we get:

$$\text{as}(f^{(\pi_i, \alpha_i)}) - \beta_i \cdot \hat{f}^{(\pi_i, \alpha_i)}(\emptyset) \leq \frac{5}{3},$$

and since $\beta_i \in \{-1, +1\}$:

$$\text{as}(f^{(\pi_i, \alpha_i)}) + |\hat{f}^{(\pi_i, \alpha_i)}(\emptyset)| \leq \frac{5}{3}.$$

Substituting $f^{(\pi_i, \alpha_i)}$ by f concludes the proof. \square

5. Discussion

In Figure 1 we summarized the most important bounds from the previous section. We plotted the average sensitivity versus the bias. Additionally we included a general lower bound on the average sensitivity as it can be found in [15]. One can see that this bounds intersects with our lower bound (which we plotted for $k = 5$), though we stated that our bound is tight. However, this is not a contradiction, since the lower bound of Theorem 2 is only achieved for highly biased functions, which are located outside the intersection.

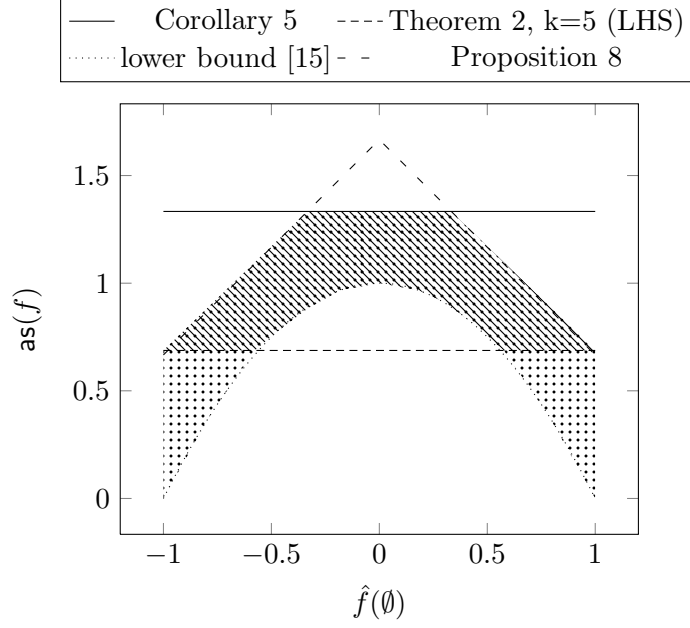


Figure 1: **Important Bounds:** The dotted-area gives the possible values for the average sensitivity of any NCF, the lined area for BF with $k = 5$ input variables.

For $k = 5$ our lower bound forms a triangle with the upper bound as formulated in Proposition 8, where the NCF with all variables being most dominant are located in the left and right corners, however, for larger k the lower bound will move further to the bottom, and with them the most dominant NCFs.

The upper bound from Corollary 5 also intersects with the bound formulated in Proposition 8. Again, this is not a contradiction, since NCF reach this bound only for small bias.

In general the average sensitivity is upper-bounded by k , i.e., $\text{as}(f) \leq k$. Further, as shown in [15], the upper-bound for unate, i.e., locally monotone, functions is $\text{as}(f) \leq \sqrt{(1 - \hat{f}(\emptyset))k}$. A function is a member of the class of unate functions, if it is monotone in each variable. That means in terms of regulatory networks that an input to a function, i.e., a regulator, can be either an activator or an inhibitor towards a certain gene. Further, NCFs form a subclass of the unate functions. Thus, the average sensitivity of NCFs is remarkably low. Since a low average sensitivity has a positive influence on the stability of Boolean networks [2], we can conjecture that networks consisting of NCF are more stable.

6. Conclusion

In this paper we investigated Boolean functions, in particular canalizing and nested canalizing functions, using Fourier analysis. We gave recursive representations for the zero coefficient and the average sensitivity based on the concept of restricted BFs.

Further, we addressed the average sensitivity for nested canalizing functions and derived and proofed a tight upper and lower bound. We show that the lower bound is achieved by functions, whose input variables are all most dominant and which are maximizing the zero coefficient and, hence, the bias. The upper bound is reached by functions, whose canalized values are alternating, and which are minimizing the bias.

We then generalized the upper bound to $\text{as}(f) \leq \frac{4}{3}$, which has been conjectured in literature, however no proof has been given so far. Finally we derived a common bound for bias and average sensitivity and discussed the stabilizing influence of the class of nested canalizing functions on the network dynamics.

It is worth noting that all these results depend on the assumption of uniform distributed inputs (see Section 2). This opens the question if the results can be generalized to other distributions. The recursive representations can be easily extended to product distributed input variables. But without further constraints there always exists a distribution which maximizes the (accordingly defined) average sensitivity, i.e., for any function with k relevant variables the average sensitivity can be k .

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A. Proof of Proposition 5

Proof. The proof for the zero and first order coefficients, i.e. $|U| = 1$ and $U = \emptyset$, follows directly from Eq. (5). We can hence use Eq. (6) as an induction hypothesis for coefficients with order smaller than $|U|$. We show next that as a result this is also valid for coefficients with order $|U| + 1$.

Since

$$\hat{f}^{(U, \bar{\alpha})}(T) = \sum_{S \subseteq U} \left(\chi_S(\bar{\alpha}) \cdot \hat{f}(T \cup S) \right),$$

and f is canalizing in any variable k , and hence every restriction of f must also be canalizing in variable k , i.e. $\hat{f}^{(U)}(\emptyset) + a_k \cdot \hat{f}^{(U)}(\{k\}) = b$, we get:

$$\begin{aligned} b &= \sum_{S \subseteq U} \left(\chi_S(\bar{\alpha}) \cdot \hat{f}(S) \right) + a_k \cdot \sum_{S \subseteq U} \left(\chi_S(\bar{\alpha}) \cdot \hat{f}(\{k\} \cup S) \right), \\ b &= \sum_{S \subseteq U, S \neq \emptyset} \left(-a_j \cdot \chi_{S \setminus \{j\}}(\bar{\alpha}) \cdot \hat{f}(S) \right) + \hat{f}(\emptyset) \\ &\quad + a_k \cdot \sum_{S \subseteq U, S \neq \emptyset} \left(\chi_S(\bar{\alpha}) \cdot \hat{f}(\{k\} \cup S) \right) + a_k \cdot \hat{f}(\{k\}). \end{aligned}$$

Using Eq. (5) and using the induction hypothesis, we get

$$\begin{aligned} 0 &= \sum_{S \subseteq U, S \neq \emptyset} (-c) + \sum_{S \subset U, S \neq \emptyset} \left(a_k \cdot \chi_S(\bar{\alpha}) \cdot \hat{f}(\{k\} \cup S) \right) \\ &\quad + \left(a_k \cdot \chi_U(\bar{\alpha}) \cdot \hat{f}(\{k\} \cup U) \right). \end{aligned}$$

We again assume (6) as true for all $S \subset U$, i.e. $|S| < |U|$ and, hence, write:

$$\begin{aligned} 0 &= -(2^{|U|} - 1) \cdot c + (2^{|U|} - 2) \cdot c + \left(a_k \cdot \chi_U(\bar{\alpha}) \cdot \hat{f}(\{k\} \cup U) \right) \\ c &= a_k \cdot \chi_U(\bar{\alpha}) \cdot \hat{f}(\{k\} \cup U), \end{aligned}$$

which concludes the proof. \square

B. Proof of Theorem 1

Proof. Starting from the definition of as as given in Eq. (10), we can fractionize the Fourier coefficients according to Proposition 3. This yields in:

$$\begin{aligned} \text{as}(f) &= \sum_{S \subseteq [n], S \neq \emptyset} \left(\frac{1}{2} \hat{f}^{(i,+)}(S \setminus \{i\}) + \frac{1}{2} (-1)^{|S \cap \{i\}|} \hat{f}^{(i,-)}(S \setminus \{i\}) \right)^2 |S|, \\ &= \frac{1}{4} \sum_{S \subseteq [n], S \neq \emptyset} \left(\left((+1)^{|S \cap \{i\}|} \hat{f}^{(i,+)}(S \setminus \{i\}) \right)^2 \right. \\ &\quad \left. + \left((-1)^{|S \cap \{i\}|} \hat{f}^{(i,-)}(S \setminus \{i\}) \right)^2 + 2 (-1)^{|S \cap \{i\}|} \hat{f}^{(i,+)}(S \setminus \{i\}) \hat{f}^{(i,-)}(S \setminus \{i\}) \right) |S| \end{aligned}$$

which leads us to:

$$\begin{aligned} \text{as}(f) &= \frac{1}{4} \sum_{S \subseteq [n], S \neq \emptyset} \left(\hat{f}^{(i,+)}(S \setminus \{i\}) \right)^2 |S| \\ &\quad + \frac{1}{4} \sum_{S \subseteq [n], S \neq \emptyset} \left(\hat{f}^{(i,-)}(S \setminus \{i\}) \right)^2 |S| \\ &\quad + \frac{1}{2} \sum_{S \subseteq [n], S \neq \emptyset} (-1)^{|S \cap \{i\}|} \hat{f}^{(i,+)}(S \setminus \{i\}) \hat{f}^{(i,-)}(S \setminus \{i\}) |S|, \end{aligned}$$

and hence to:

$$\begin{aligned} \text{as}(f) &= \frac{1}{4} \sum_{S \subseteq [n] \setminus \{i\}, S \neq \emptyset} \left(\hat{f}^{(i,+)}(S) \right)^2 |S| + \frac{1}{4} \sum_{S \subseteq [n] \setminus \{i\}} \left(\hat{f}^{(i,+)}(S) \right)^2 (1 + |S|) \\ &\quad + \frac{1}{4} \sum_{S \subseteq [n] \setminus \{i\}, S \neq \emptyset} \left(\hat{f}^{(i,-)}(S) \right)^2 |S| + \frac{1}{4} \sum_{S \subseteq [n] \setminus \{i\}} \left(\hat{f}^{(i,-)}(S) \right)^2 (1 + |S|) \\ &\quad + \frac{1}{2} \sum_{S \subseteq [n] \setminus \{i\}, S \neq \emptyset} (-1)^0 \hat{f}^{(i,+)}(S) \hat{f}^{(i,-)}(S) |S| \\ &\quad + \frac{1}{2} \sum_{S \subseteq [n] \setminus \{i\}} (-1)^1 \hat{f}^{(i,+)}(S) \hat{f}^{(i,-)}(S) (1 + |S|). \end{aligned}$$

Since $f^{(i,a)}(S) = 0$ for all $S : i \in S$ we can write

$$\begin{aligned} \text{as}(f) &= \frac{1}{2} \underbrace{\sum_{S \subseteq [n], S \neq \emptyset} \left(\hat{f}^{(i,+)}(S) \right)^2 |S|}_{=\text{as}(f^{(i,+)})} + \frac{1}{4} \underbrace{\sum_{S \subseteq [n]} \left(\hat{f}^{(i,+)}(S) \right)^2}_{=1} \\ &\quad + \frac{1}{2} \underbrace{\sum_{S \subseteq [n], S \neq \emptyset} \left(\hat{f}^{(i,-)}(S) \right)^2 |S|}_{=\text{as}(f^{(i,-)})} + \frac{1}{4} \underbrace{\sum_{S \subseteq [n]} \left(\hat{f}^{(i,-)}(S) \right)^2}_{=1} \\ &\quad + \frac{1}{2} \sum_{S \subseteq [n], S \neq \emptyset} \hat{f}^{(i,+)}(S) \hat{f}^{(i,-)}(S) |S| - \frac{1}{2} \sum_{S \subseteq [n], S \neq \emptyset} \hat{f}^{(i,+)}(S) \hat{f}^{(i,-)}(S) |S| \\ &\quad - \frac{1}{2} \sum_{S \subseteq [n]} \hat{f}^{(i,+)}(S) \hat{f}^{(i,-)}(S). \end{aligned}$$

Finally we get

$$\text{as}(f) = \frac{1}{2} \text{as}(f^{(i,+)}) + \frac{1}{2} \text{as}(f^{(i,-)}) + \frac{1}{2} - \frac{1}{2} \sum_{S \subseteq [n]} \hat{f}^{(i,+)}(S) \hat{f}^{(i,-)}(S),$$

which concludes the proof. \square

C. Proof of Theorem 2

Proof. We first prove the right hand side. Let us recall Corollary 4:

$$\text{as}(f) = \frac{1}{2} \left(\text{as}(f^{(\pi_i, \alpha_i)}) + 1 - \hat{f}^{(\pi_i, \alpha_i)}(\emptyset) \beta_i \right).$$

If we apply it again and use Proposition 6, we get:

$$\begin{aligned} \text{as}(f) &= \frac{1}{2} \left(\frac{1}{2} \left(\text{as}(f^{(\pi_i, \alpha_i)^{(\pi_{i+1}, \alpha_{i+1})}}) + 1 - \hat{f}^{(\pi_i, \alpha_i)^{(\pi_{i+1}, \alpha_{i+1})}}(\emptyset) \beta_{i+1} \right) \right. \\ &\quad \left. + 1 - \left(\frac{1}{2} \hat{f}^{(\pi_i, \alpha_i)^{(\pi_{i+1}, \alpha_{i+1})}}(\emptyset) + \frac{1}{2} \beta_{i+1} \right) \beta_i \right) \\ &= \frac{1}{4} \text{as}(f^{(\pi_i, \alpha_i)^{(\pi_{i+1}, \alpha_{i+1})}}) + \frac{3}{4} - \frac{1}{4} \hat{f}^{(\pi_i, \alpha_i)^{(\pi_{i+1}, \alpha_{i+1})}}(\emptyset) \beta_{i+1} \\ &\quad - \frac{1}{4} \hat{f}^{(\pi_i, \alpha_i)^{(\pi_{i+1}, \alpha_{i+1})}}(\emptyset) \beta_i - \frac{1}{4} \beta_{i+1} \beta_i \\ &= \frac{1}{4} \text{as}(f^{(\pi_i, \alpha_i)^{(\pi_{i+1}, \alpha_{i+1})}}) - \frac{1}{4} \hat{f}^{(\pi_i, \alpha_i)^{(\pi_{i+1}, \alpha_{i+1})}}(\emptyset) (\beta_i + \beta_{i+1}) \\ &\quad - \frac{1}{4} \beta_{i+1} \beta_i + \frac{3}{4} \end{aligned}$$

Since $\beta_i, \beta_{i+1} \in \{-1, +1\}$ and $|\hat{f}^{(\pi_i, \alpha_i)^{(\pi_{i+1}, \alpha_{i+1})}}(\emptyset)| \leq 1$, we can upper-bound $-\frac{1}{4} \hat{f}^{(\pi_i, \alpha_i)^{(\pi_{i+1}, \alpha_{i+1})}}(\emptyset) (\beta_i + \beta_{i+1}) - \frac{1}{4} \beta_{i+1} \beta_i$ as:

$$-\frac{1}{4} \hat{f}^{(\pi_i, \alpha_i)^{(\pi_{i+1}, \alpha_{i+1})}}(\emptyset) (\beta_i + \beta_{i+1}) - \frac{1}{4} \beta_{i+1} \beta_i \leq \frac{1}{4}$$

and finally upper-bound $\text{as}(f)$ as

$$\text{as}(f) \leq \frac{1}{4} \text{as}(f^{(\pi_i, \alpha_i)^{(\pi_{i+1}, \alpha_{i+1})}}) + 1, \quad (16)$$

where $f^{(\pi_i, \alpha_i)^{(\pi_{i+1}, \alpha_{i+1})}}$ has $k-2$ relevant variables. We will now show the theorem by induction. For $k=1$ Eq. (13) simplifies to

$$\text{as}(f) \leq 1,$$

which is obviously true. For $k=2$ Eq. (13) results in

$$\text{as}(f) \leq 1,$$

which is also true and can be verified by inspecting all possible functions.

Using Eq. (13) as the induction hypothesis, we will now show that Eq. (13) is true for k if it is true for $k - 2$. Using Eq. 16, we get:

$$\text{as}(f) \leq \frac{1}{4} \text{as}(f^{(\pi_i, \alpha_i)(\pi_{i+1}, \alpha_{i+1})}) + 1$$

since $f^{(\pi_i, \alpha_i)(\pi_{i+1}, \alpha_{i+1})}$ has $k - 2$ relevant variables, we can use our hypothesis and write

$$\begin{aligned} \text{as}(f) &\leq \frac{1}{4} \left(\frac{4}{3} - 2^{-(k-2)} - \frac{1}{3} \cdot 2^{-(k-2)} \cdot (-1)^{k-2} \right) + 1 \\ &= \frac{4}{3} - 2^{-k} - \frac{1}{3} 2^{-k} (-1)^k, \end{aligned}$$

which concludes the induction.

The left hand side is commonly known in literature and can be proven along the lines like the right hand side using the following inequality, which follows from Corollary 4 and Proposition 7:

$$\text{as}(f) \geq \frac{1}{2} \left(\text{as}(f^{(\pi_i, \alpha_i)}) + \frac{1}{2^{k-2}} \right).$$

□

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